Aspects of degeneracy in the complex-energy plane

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GANIL - ORNL Theory Collaboration

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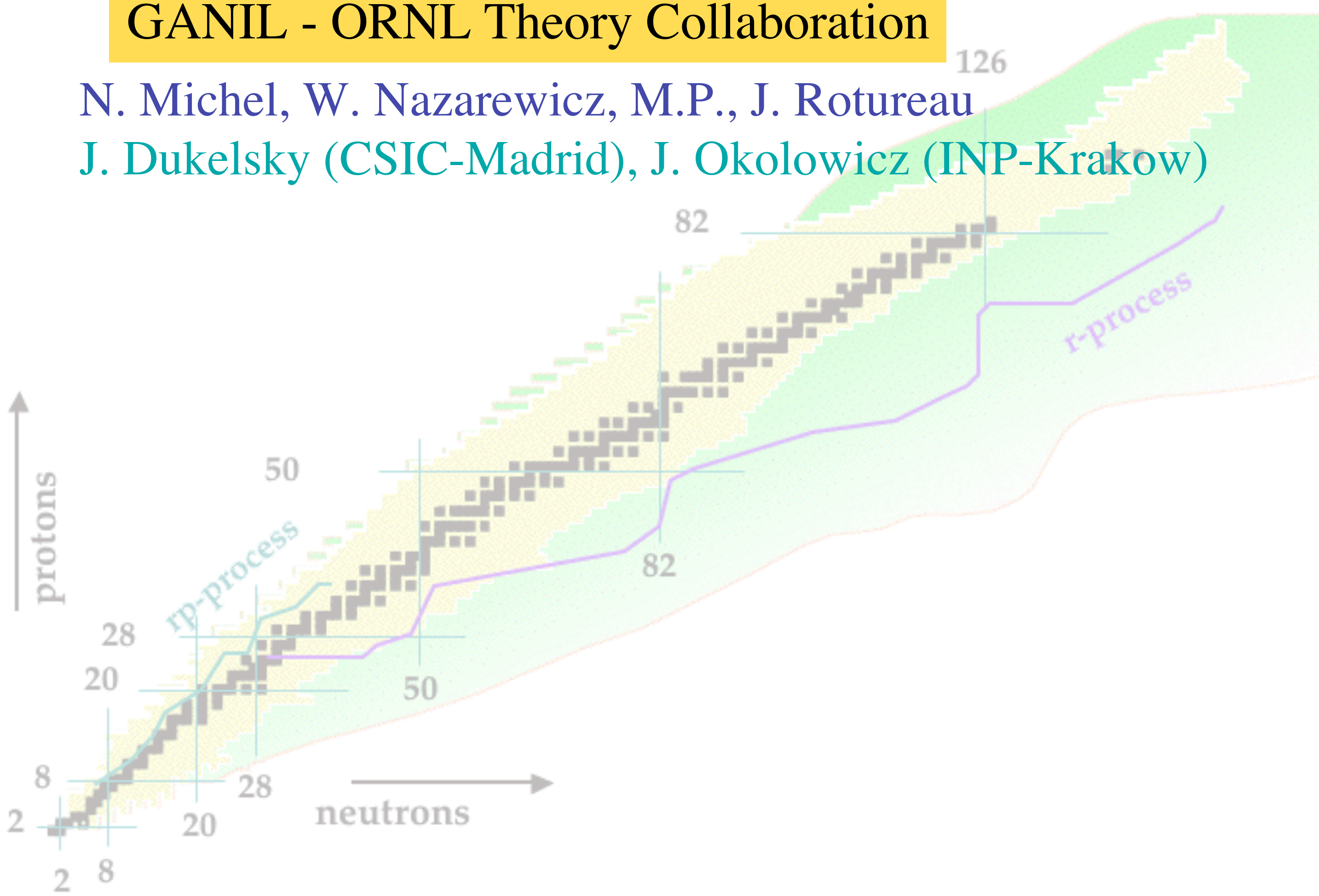
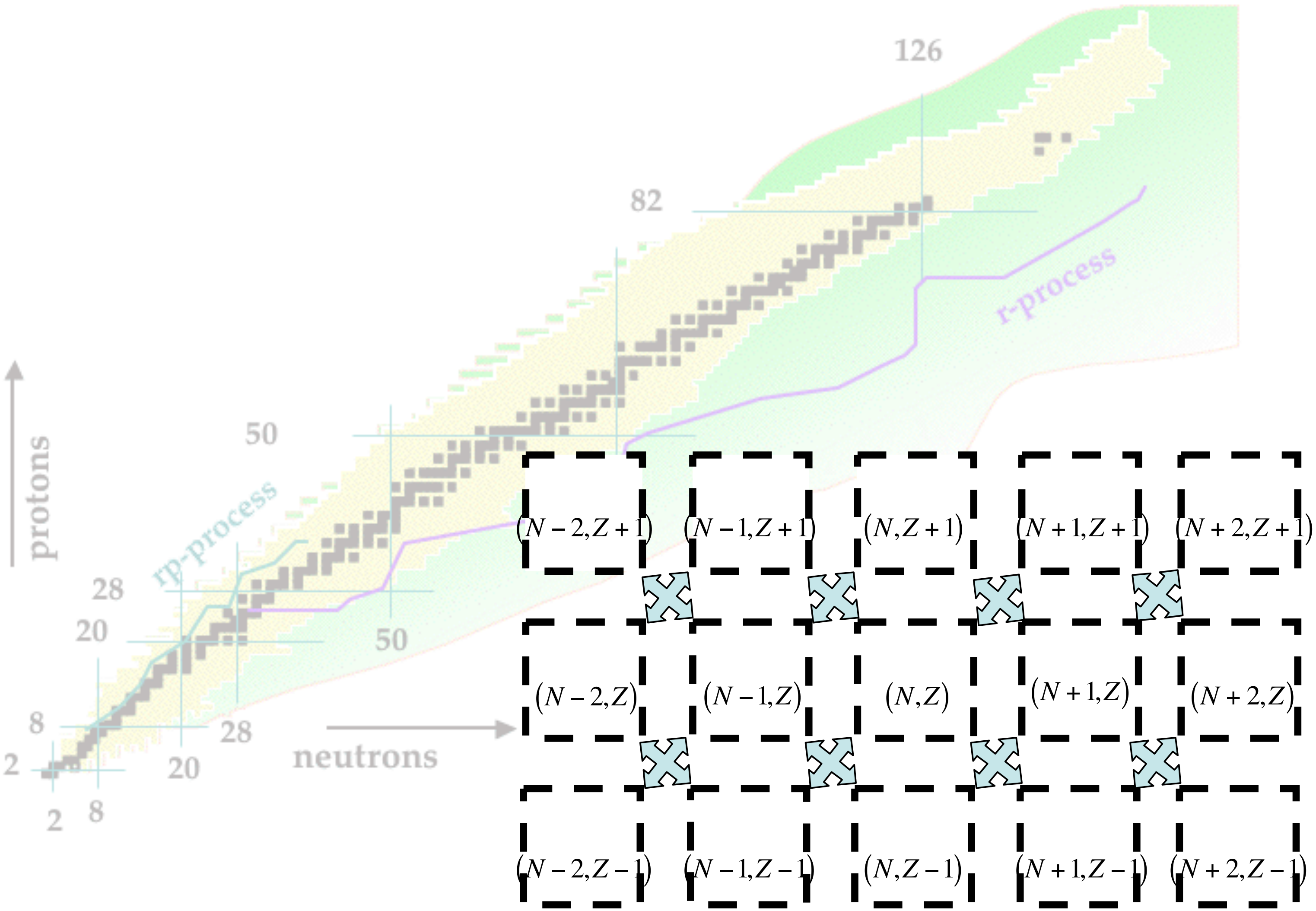
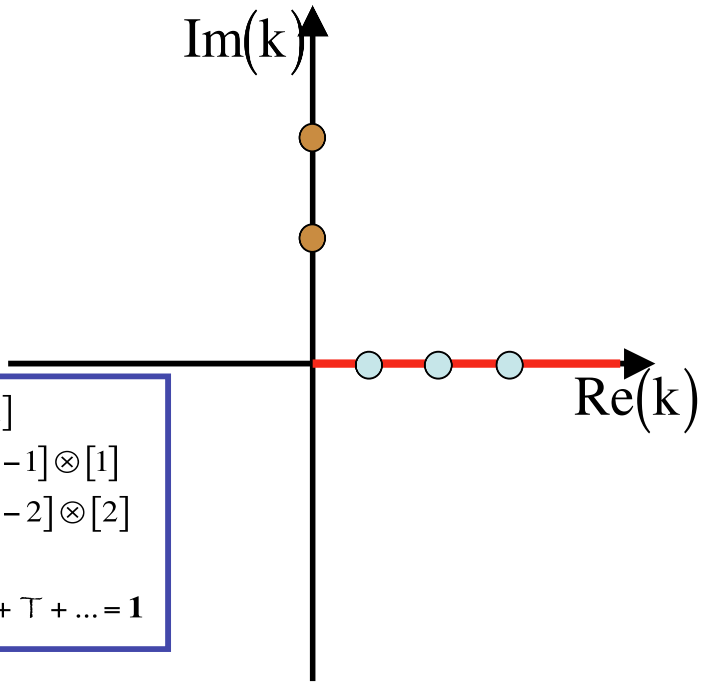



126
82
50
28
20
8
2
2
8
20
28
50
protons
neutrons
rp-process
r-process
(N − 2, Z + 1)
(N − 1, Z + 1)
(N, Z + 1)
(N + 1, Z + 1)
(N + 2, Z + 1)
(N − 2, Z)
(N − 1, Z)
(N, Z)
(N + 1, Z)
(N + 2, Z)
(N − 2, Z − 1)
(N − 1, Z − 1)
(N, Z − 1)
(N + 1, Z − 1)
(N + 2, Z − 1)

QM in rigged Hilbert space ⟶ complex-energy continuum SM (Gamow SM)

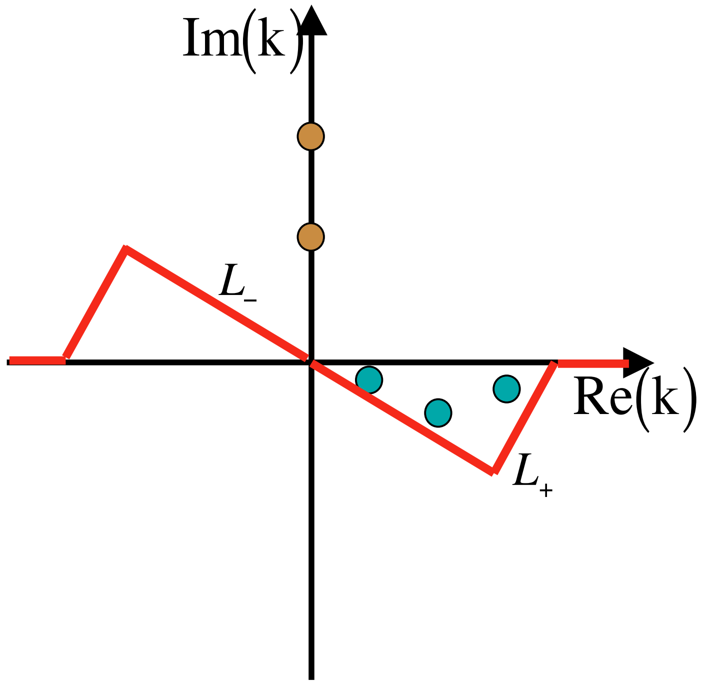
QM in Hilbert space using projected subspaces ⟶ real-energy continuum SM (SMEC)

Hilbert space



$$\mathcal{Q} = [A]$$
$$\mathcal{P} = [A-1] \otimes [1]$$
$$\mathcal{T} = [A-2] \otimes [2]$$
$$\ldots = \ldots$$
$$\mathcal{Q} + \mathcal{P} + \mathcal{T} + \ldots = \mathbf{1}$$

rigged Hilbert space



$$H_{QQ} \rightarrow \mathcal{H}_{QQ} = H'_{QQ} - \frac{i}{2} V V^T$$

$[M \times M]$ $[M \times \Lambda]$

$$\mathcal{H}_{QQ} \rightarrow [\mathcal{H}_{QQ}]_{ij} = [\mathcal{H}_{QQ}]_{ji} \quad \longleftrightarrow \quad H \rightarrow [H]_{ij} = [H]_{ji}$$

One obtains non-Hermitian eigenvalue problem: $[\mathcal{H}_{QQ}]_{ij} \neq [\mathcal{H}^{+}_{QQ}]_{ij}, [H]_{ij} \neq [H]^{+}_{ij}$

Hierarchy of structures in quantal spectra

M.V.Berry (1983)

- degeneracies $\delta E = 0$
- splittings due to barrier penetration $\delta E \sim \mathcal{O}\left(\exp\left\{-\hbar^{-1}\right\}\right)$
- …
- oscillatory clusterings of levels (closed classical orbits) $\delta E \sim \mathcal{O}(\hbar)$

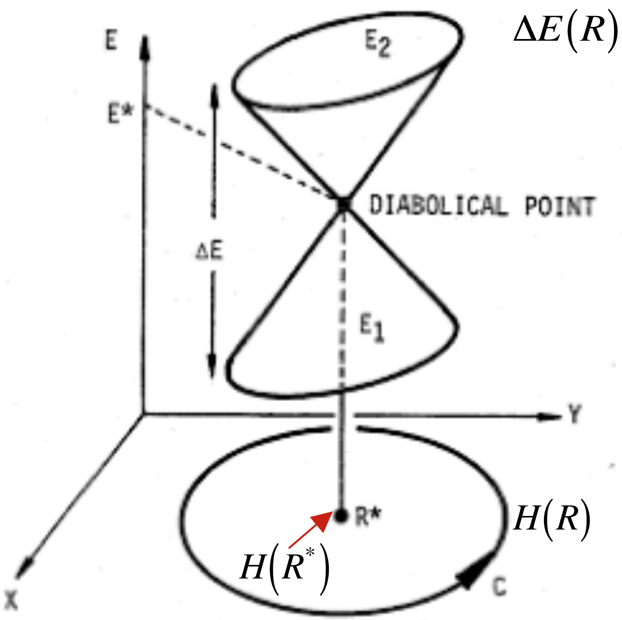
Without symmetry, degeneracies are considered to be ‘accidental’

How accidental is a degeneracy? J.von Neumann, E.Wigner (1929)

Consider $H\left(R^*\right) \subset \left\{H(R)\right\}$,$(R \equiv X, Y, \ldots)$, $H(R)$ - Hermitian Hamiltonians

$$\Delta E(R) \equiv E_2(R) - E_1(R) = \left\{\left[H'_{11}(R) - H'_{22}(R)\right]^2 + 4\left|H'_{12}(R)\right|^2\right\}^{1/2}$$

$$\Leftrightarrow \Delta E(X,Y) = \left\{A\left(X - X^*\right)^2 + 2B\left(X - X^*\right)\left(Y - Y^*\right) + C\left(Y - Y^*\right)^2\right\}^{1/2} = 0$$

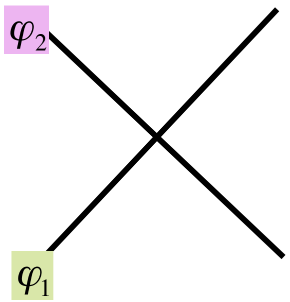


$$\Phi_n(C) = \Phi_n e^{\pm i\Omega(C)/2}$$

Molecular physics:
M.Vaz Pires et al., J. Chem Phys. 69 (1978) 3242
Nuclear physics:
D.L.Hill, J.A.Wheeler, Phys. Rev. 89 (1952) 1102

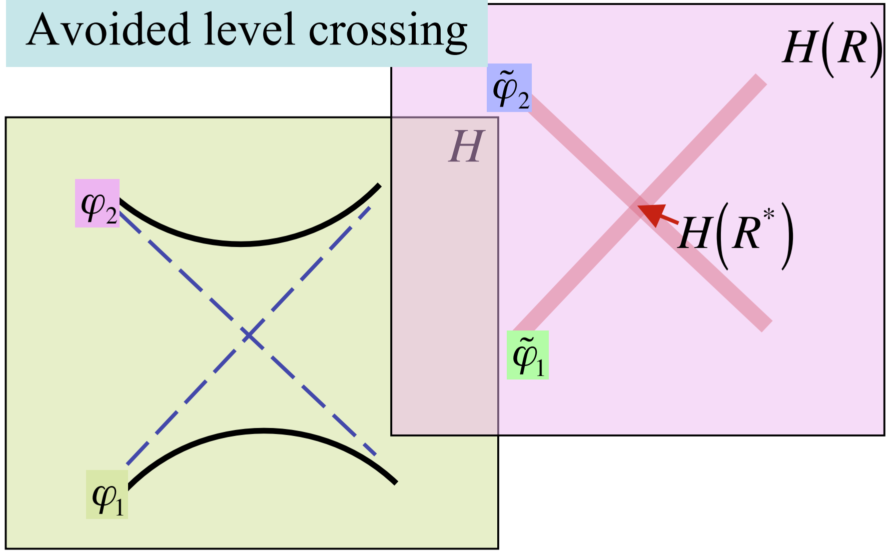
Hermitian problems

Sharp level crossing



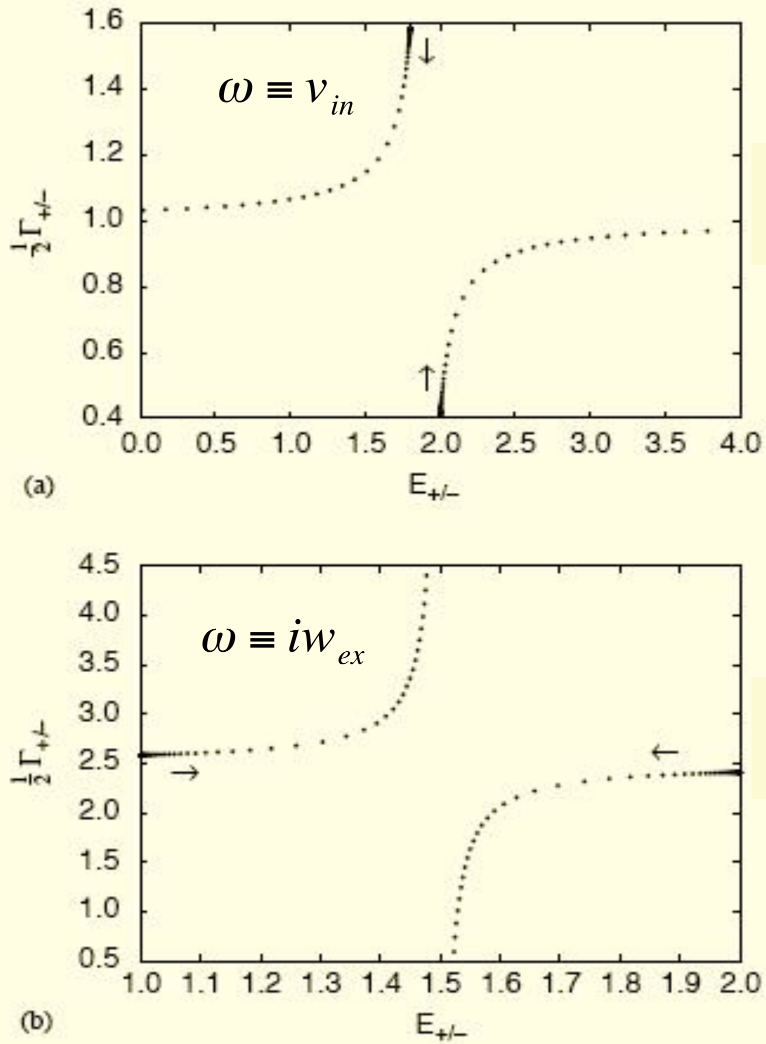
- symmetry
- integrable system

Avoided level crossing



- non-integrable system

Non-Hermitian problems



Level repulsion
Width clustering

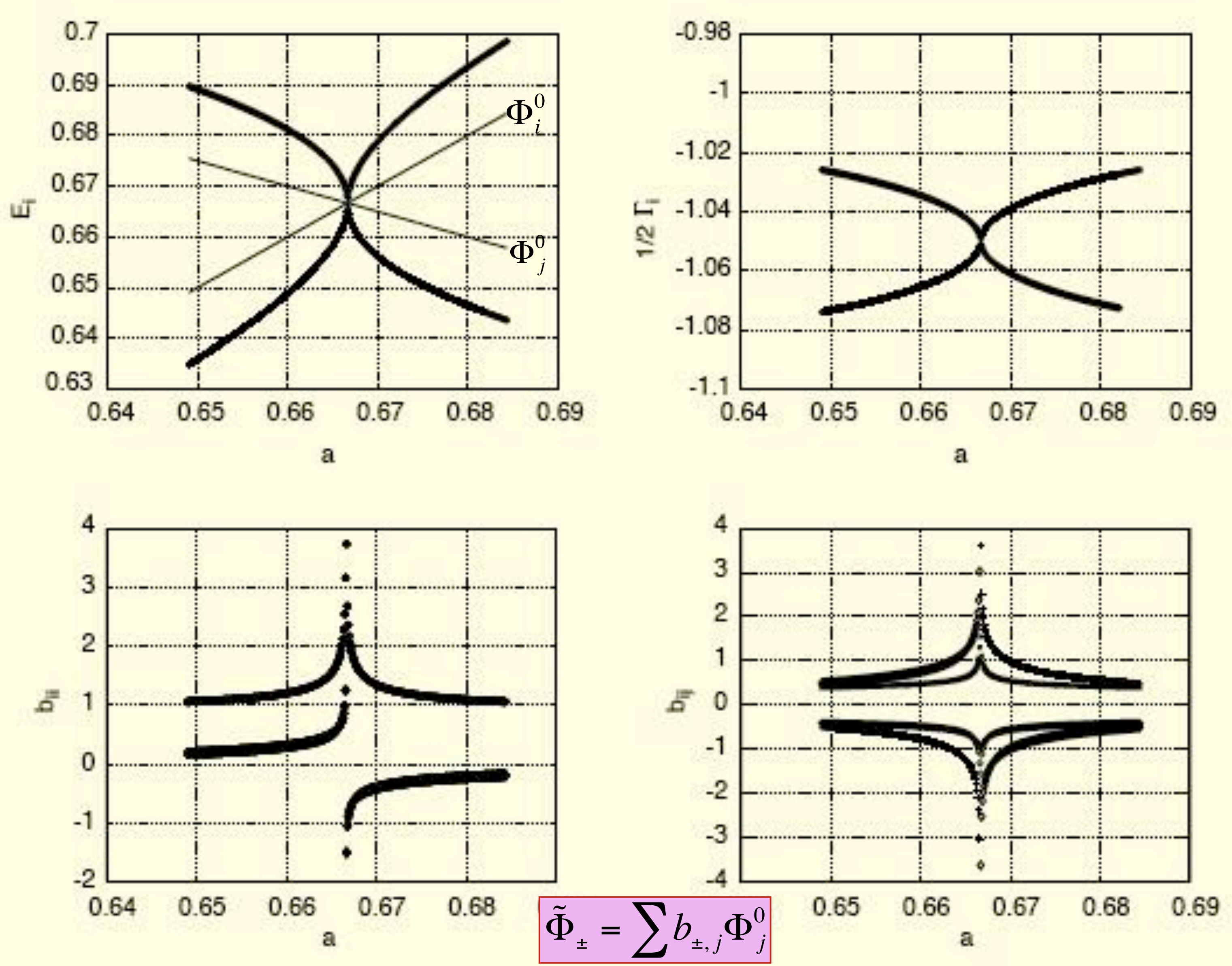
Level clustering
Width repulsion

$$\mathscr{H} = \begin{pmatrix} \epsilon_1 & \omega \\ \omega & \epsilon_2 \end{pmatrix} \equiv \begin{pmatrix} e_1 - \frac{i}{2}\gamma_1 & 0 \\ 0 & e_2 - \frac{i}{2}\gamma_2 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$$

$\omega = v_{in} + i w_{ex}$

$$\mathscr{E}_\pm \equiv E_\pm^{(v)} - \frac{i}{2}\Gamma_\pm^{(v)} = \frac{\epsilon_1 + \epsilon_2}{2} \pm \frac{1}{2}\sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\omega^2}$$

Exceptional (branching) point



$$\tilde{\Phi}_{\pm} = \sum b_{\pm,j} \Phi_j^0$$

$e_1 = 1 - a/2$

$e_2 = a$

$\gamma_1/2 = 1$

$\gamma_2/2 = 1.1$

3-level pairing model

ε_3 ——— Ω_3

ε_2 ——— Ω_2

ε_1 ——— Ω_1

$$\mathcal{H}_{eff} = \underbrace{\sum_i \varepsilon_i N_i + g \sum_{ij} A_i^\dagger A_j}_{\text{integrable}} - \boxed{g' \sum_i N_i^2}$$

g, g' - complex

$$N_l = \sum_m a_{lm}^\dagger a_{lm} \qquad A_l^\dagger = \sum_m a_{lm}^\dagger a_{l\bar{m}}^\dagger$$

$\dim_H = n$

Degeneracies in the spectrum:

$$\left.\begin{array}{l} P(E,g) \equiv \det\left[E - H(g)\right] = 0 \\ \dfrac{\partial}{\partial E} P(E,g) = 0 \end{array}\right\} D(g) = 0 \Rightarrow g_1, \dots, g_{N_D} \quad \text{(positions of degeneracies)}$$

$$\max(N_D) = n(n-1)/2$$

For integrable systems: $N_D < n(n-1)/2$

The third integral of motion:
$$Q(G) = \left[1 + g\left(\frac{\Omega_2}{\varepsilon_2} + \frac{\Omega_3}{\varepsilon_3}\right)\right]\frac{N_1}{2} + \frac{g\Omega_1}{\varepsilon_2}\frac{N_2}{2} + \frac{g\Omega_1}{\varepsilon_3}\frac{N_3}{2} -$$
$$-g\left\{\frac{1}{\varepsilon_2}\left[\frac{1}{2}\left(A_1^\dagger A_2 + A_2^\dagger A_1\right) + N_1 N_2\right] + \frac{1}{\varepsilon_3}\left[\frac{1}{2}\left(A_1^\dagger A_3 + A_3^\dagger A_1\right) + N_1 N_3\right]\right\}$$

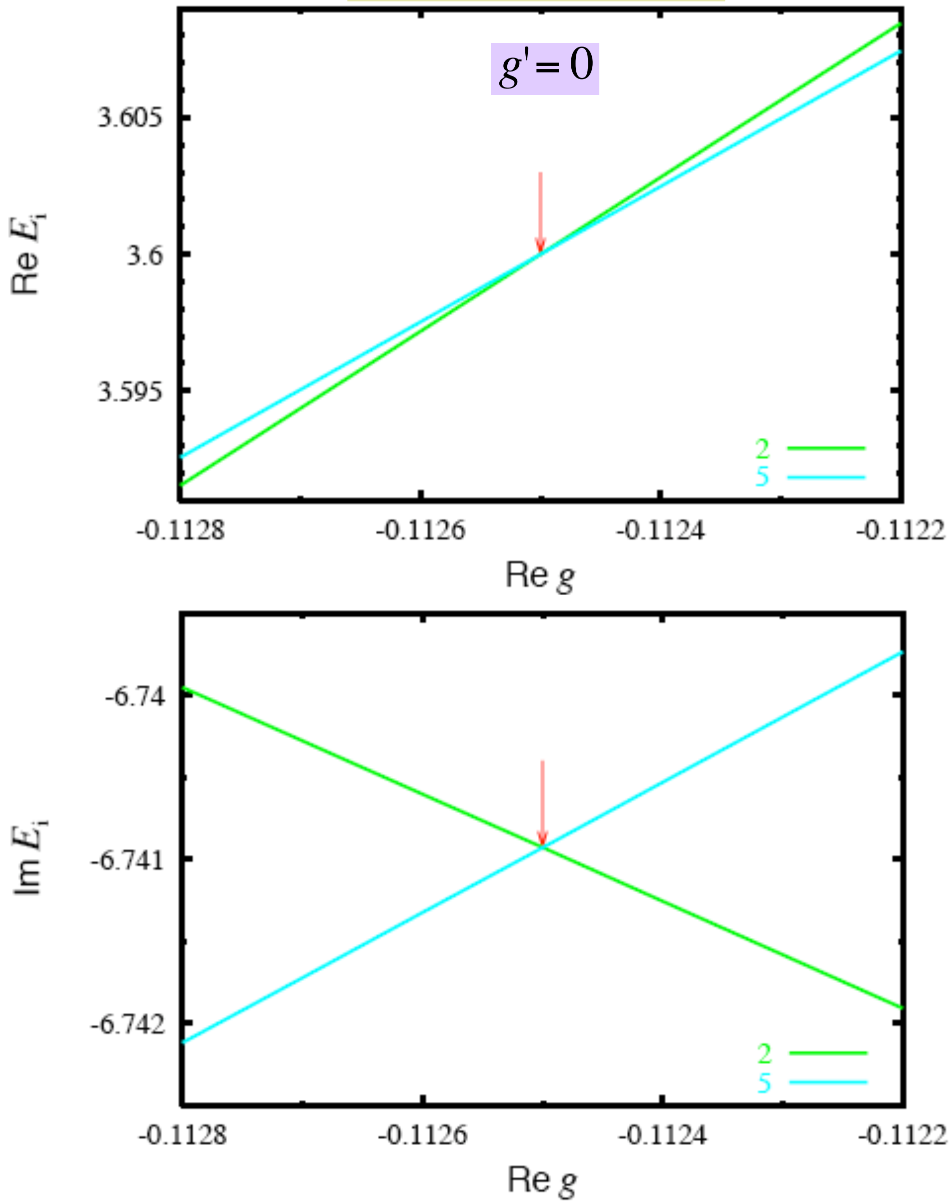
Integrable, non-Hermitian systems

$$g' = 0;\ \mathcal{H}_{eff}\left(\varepsilon_i, g\right)$$

$$N_D < n\left(n-1\right)/2$$

Non-singular resonance crossing (double-root)

Energy eigenvalues

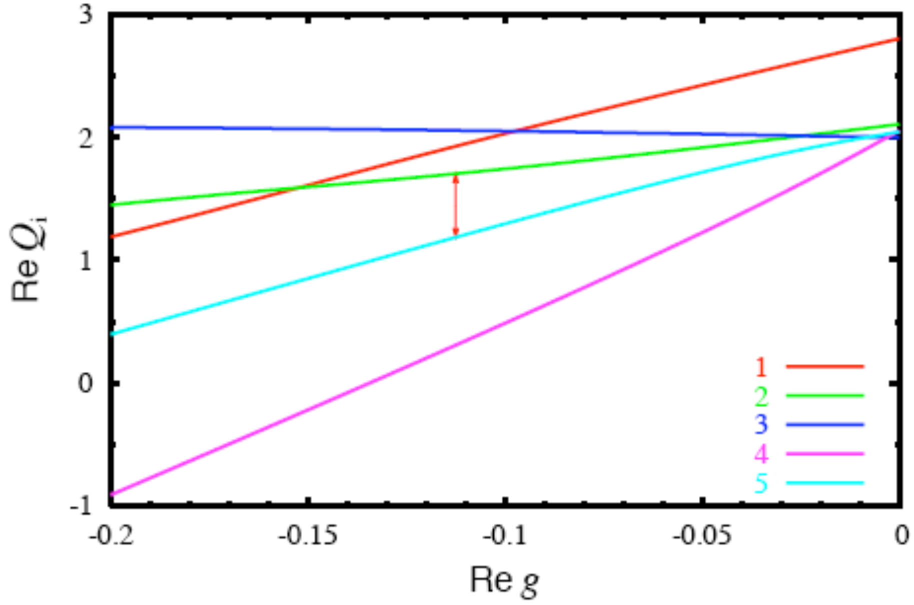


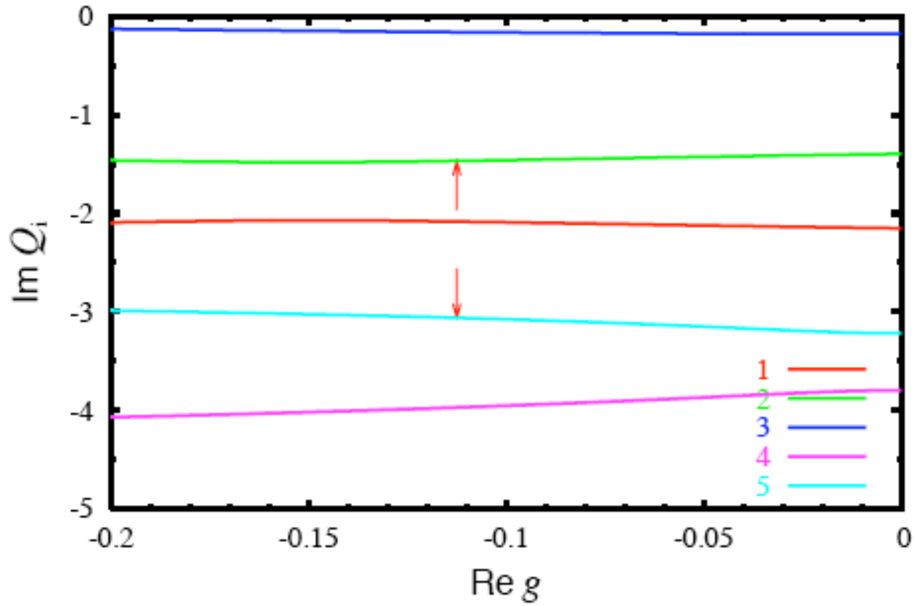
$|i_1\rangle \neq |i_2\rangle$

$\dim_H(g_i) = n$

$$Q_i \equiv \langle i|\hat{Q}|i\rangle$$

$$Q_{i_1}(g_i) \neq Q_{i_2}(g_i)$$

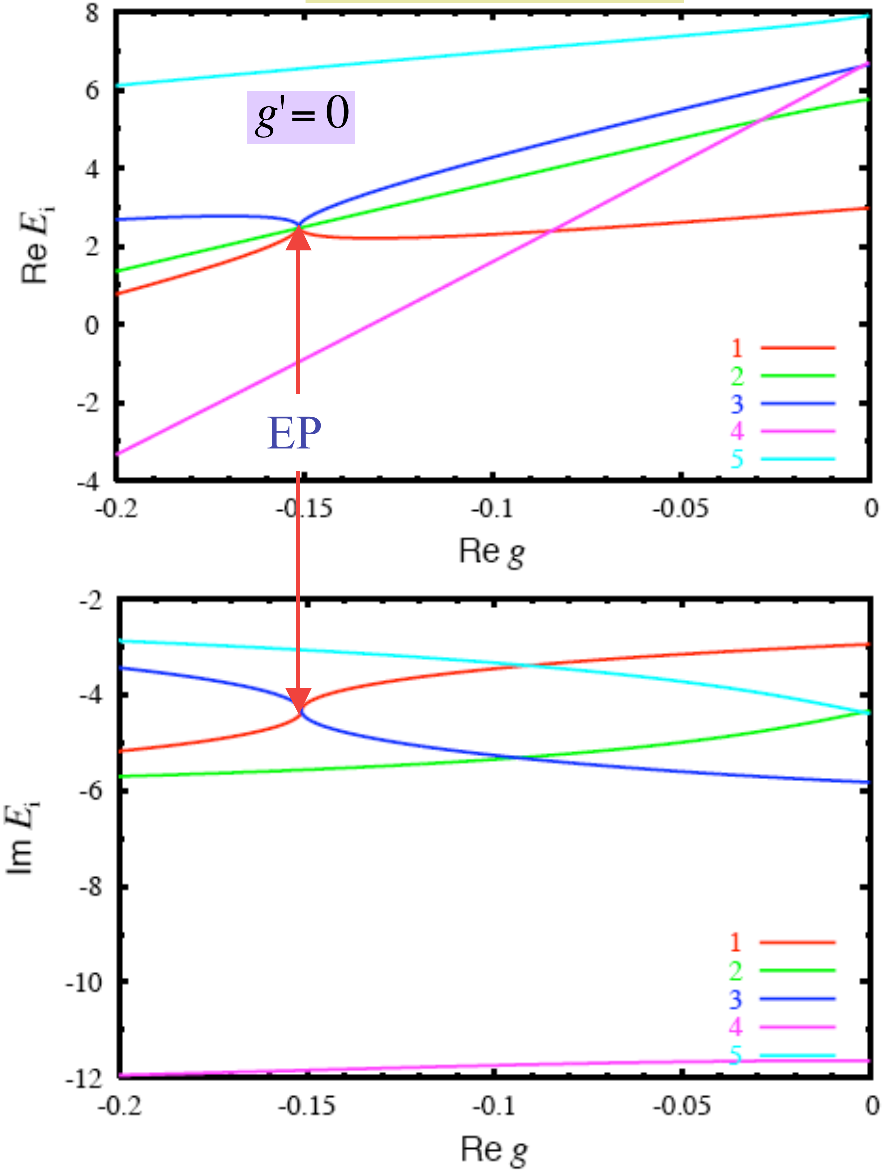




Regular behavior of $\langle i_1|\hat{O}|i_1\rangle, \langle i_2|\hat{O}|i_2\rangle$ at the crossing

Exceptional (branching) point

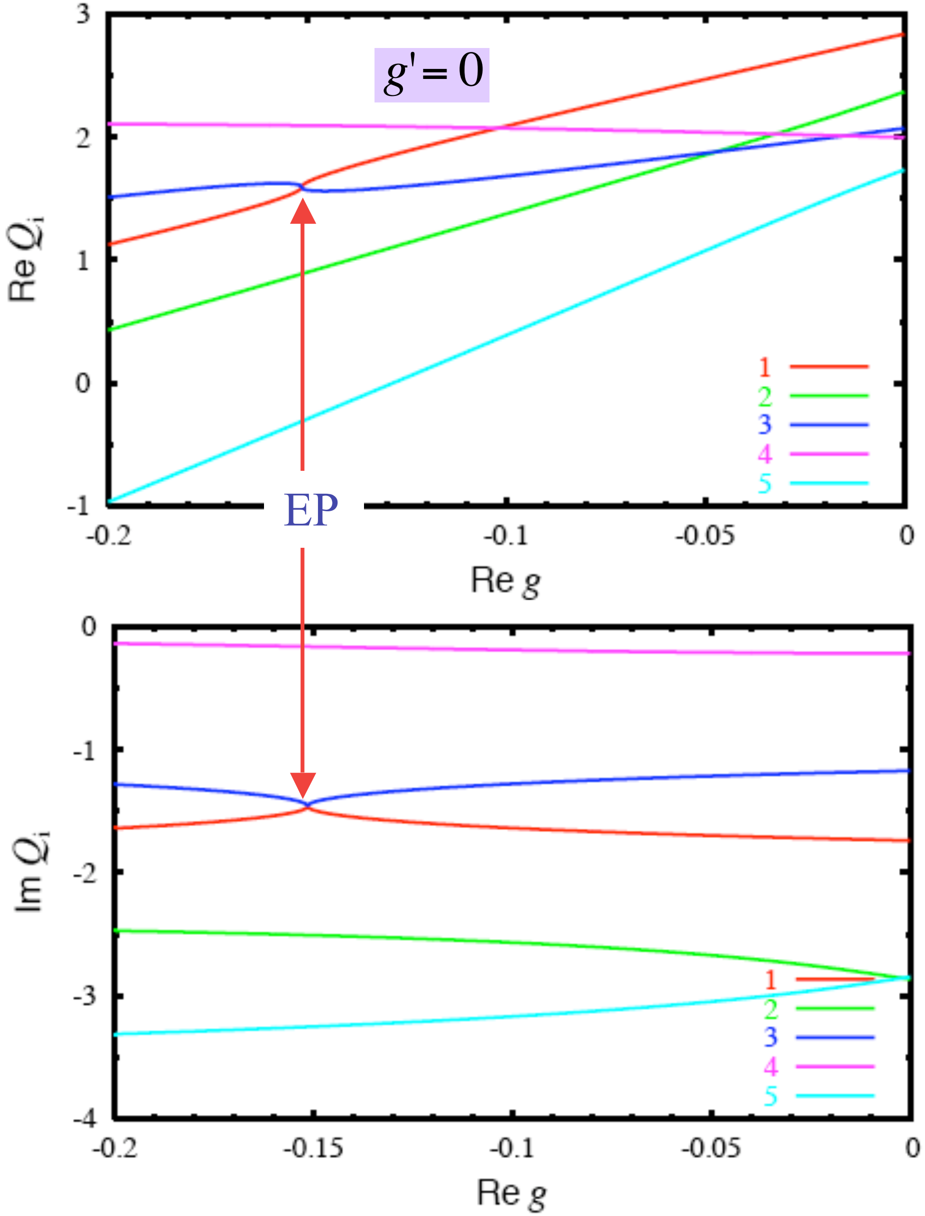
Energy eigenvalues



$|i_1\rangle \neq |i_2\rangle$

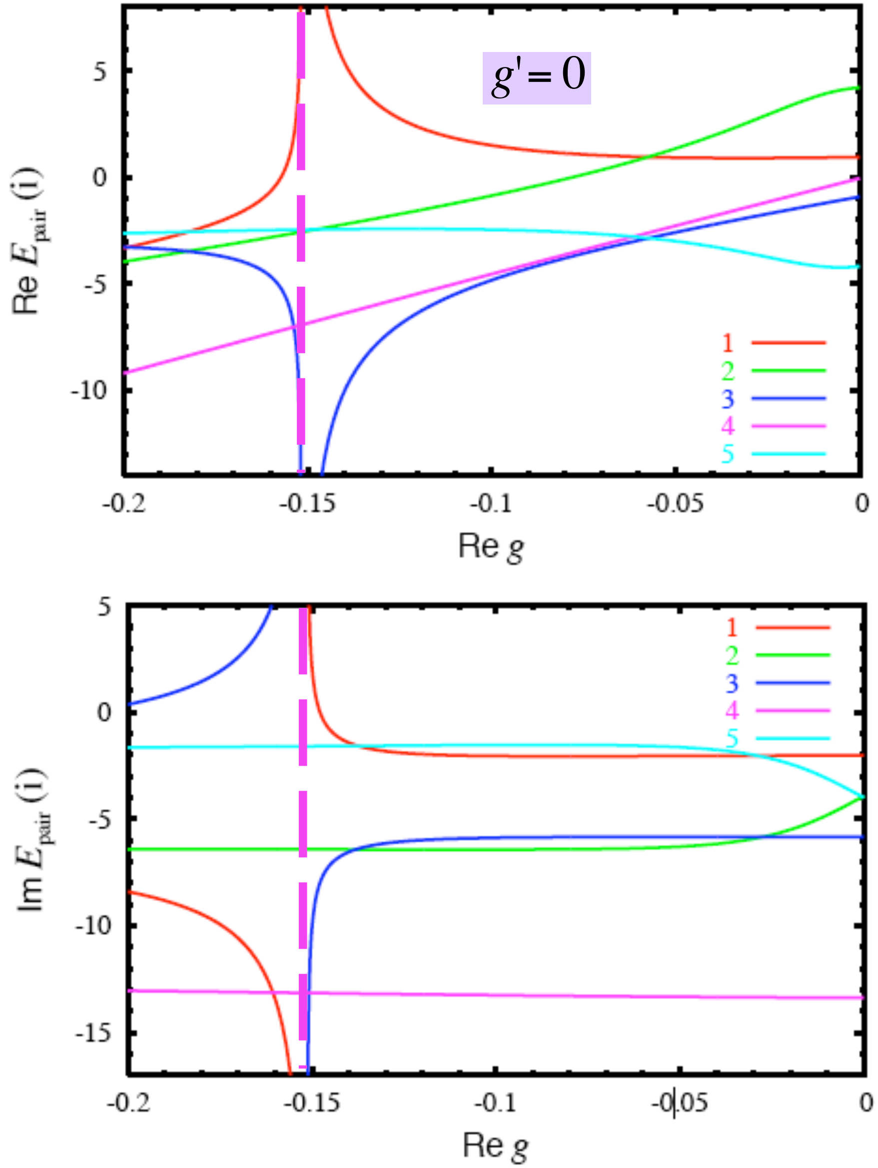
$\dim_H(g_i) = n$

$$Q_i \equiv \langle i|\hat{Q}|i\rangle$$



$$Q_{i_1}(g_i) = Q_{i_2}(g_i)$$

$$E_{\text{pair}}(i) \equiv \langle i | g \sum_{kl} A_k^+ A_l | i \rangle$$



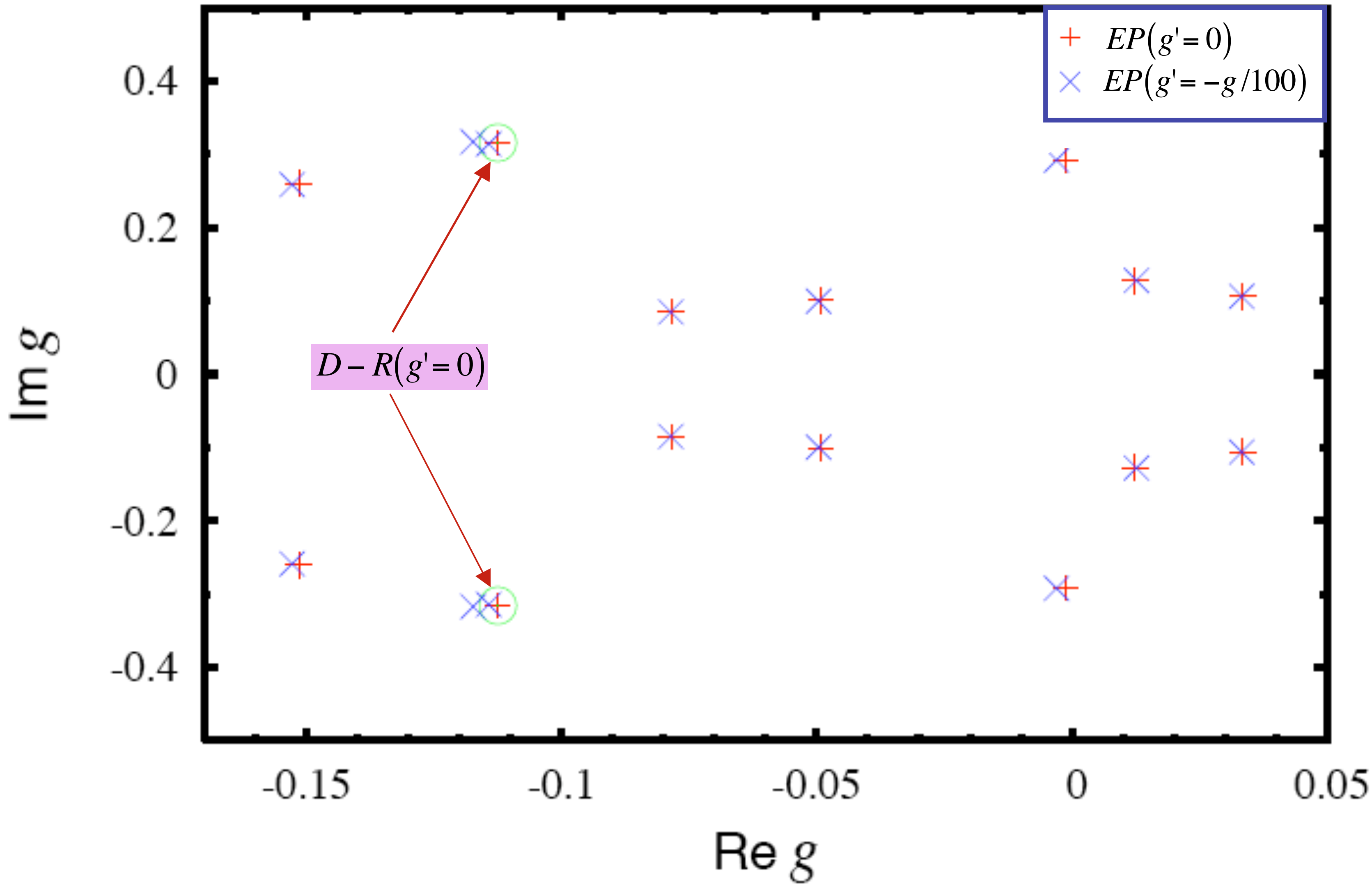
Singular behavior of $\langle i_1 | \hat{O} | i_1 \rangle, \langle i_2 | \hat{O} | i_2 \rangle$; Regular behavior of $\langle i_1 | \hat{O} | i_1 \rangle + \langle i_2 | \hat{O} | i_2 \rangle$

Non-integrable, non-Hermitian systems

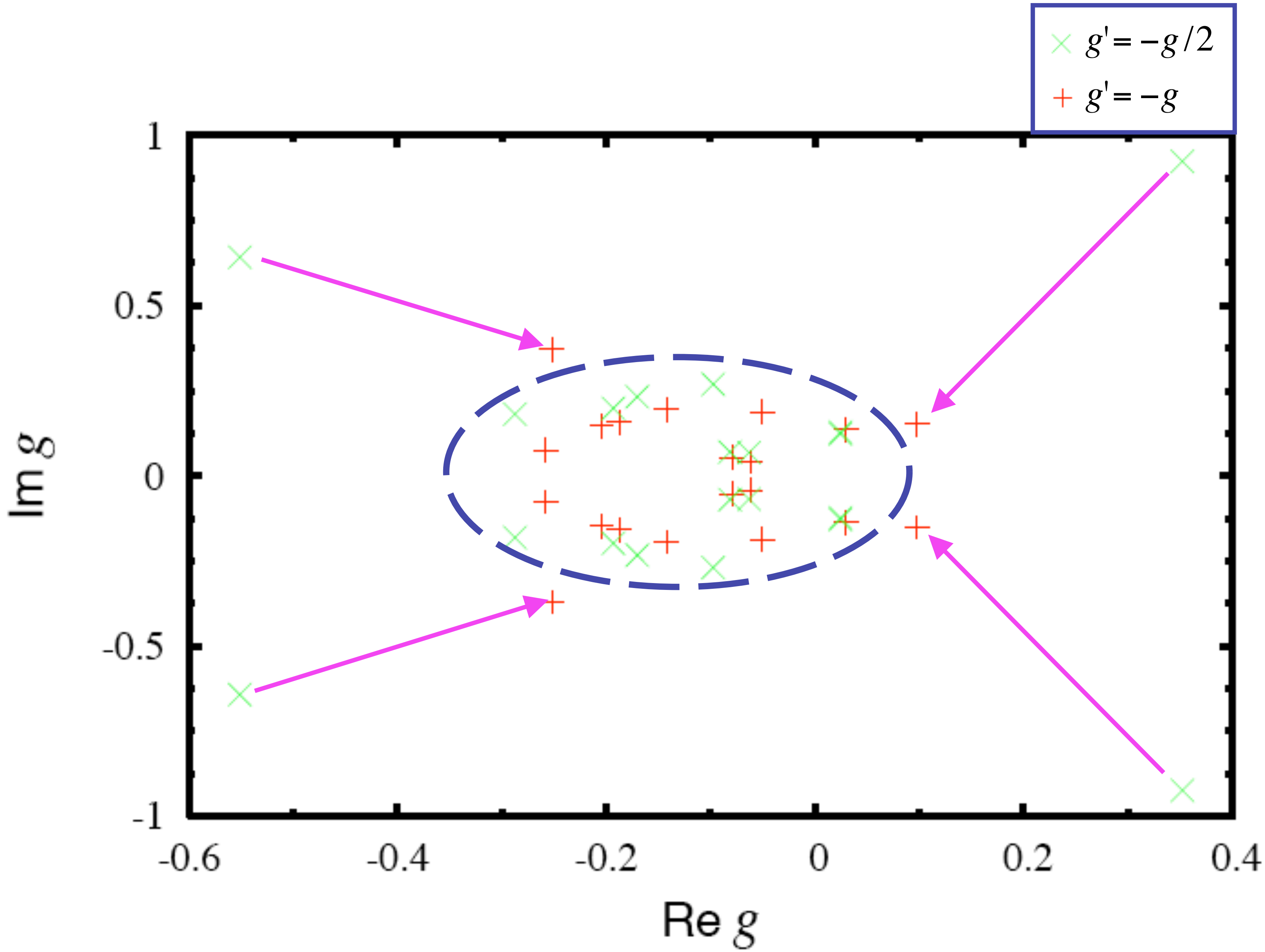
$$g' \neq 0; g' = \varphi(g); \mathcal{H}_{eff}(\varepsilon_i, g)$$

$$N_D = n(n-1)/2$$

Quasi-integrable systems



Non-integrable perturbation breaks each D-R into the two EPs

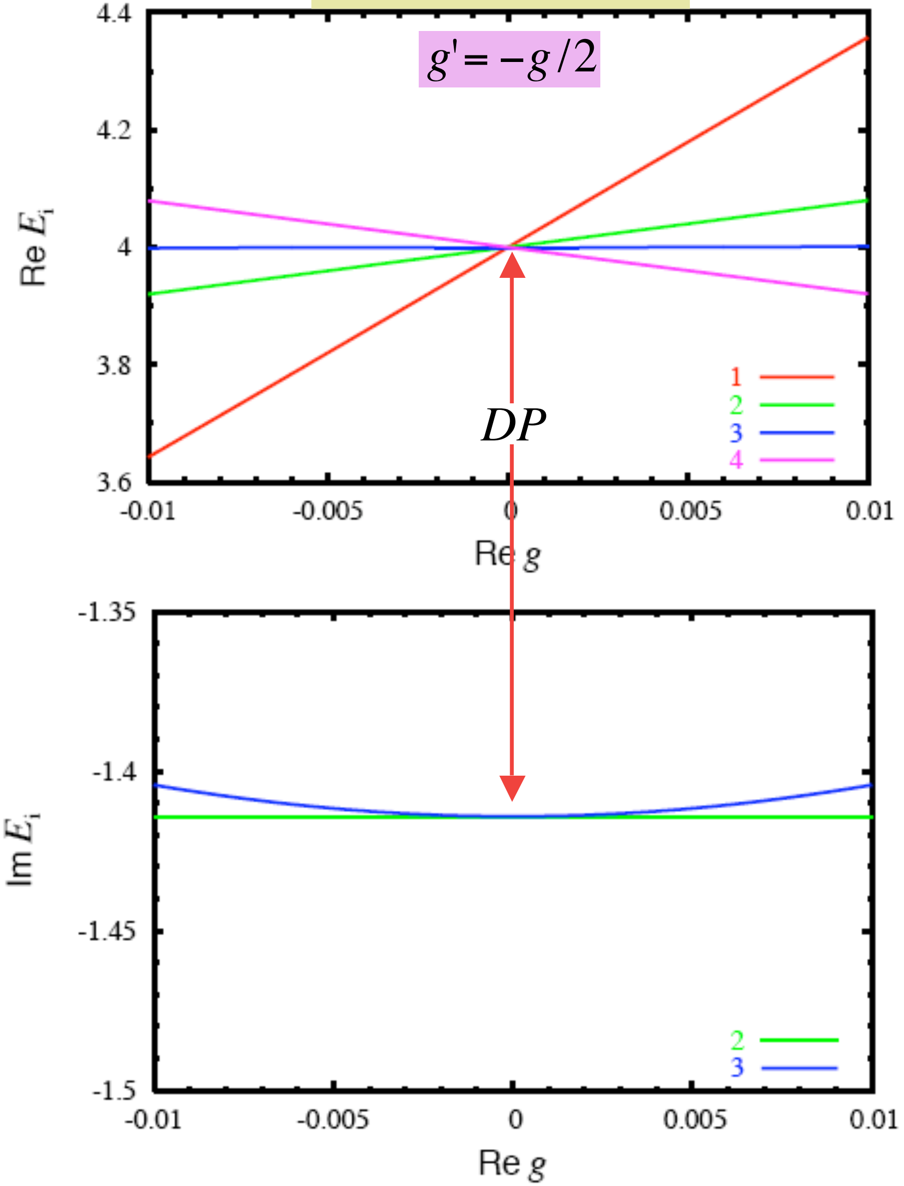


‘Missing roots’ appear for finite g-values

Diabolic point $(g_i = g_{i+1})$

Energy eigenvalues

$g' = -g/2$



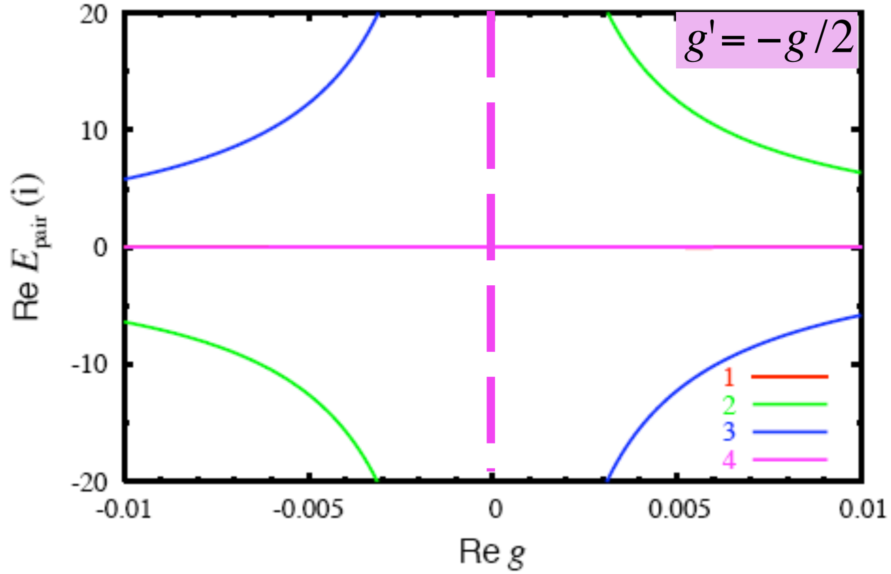
$|i_1\rangle \neq |i_2\rangle = |0\rangle$

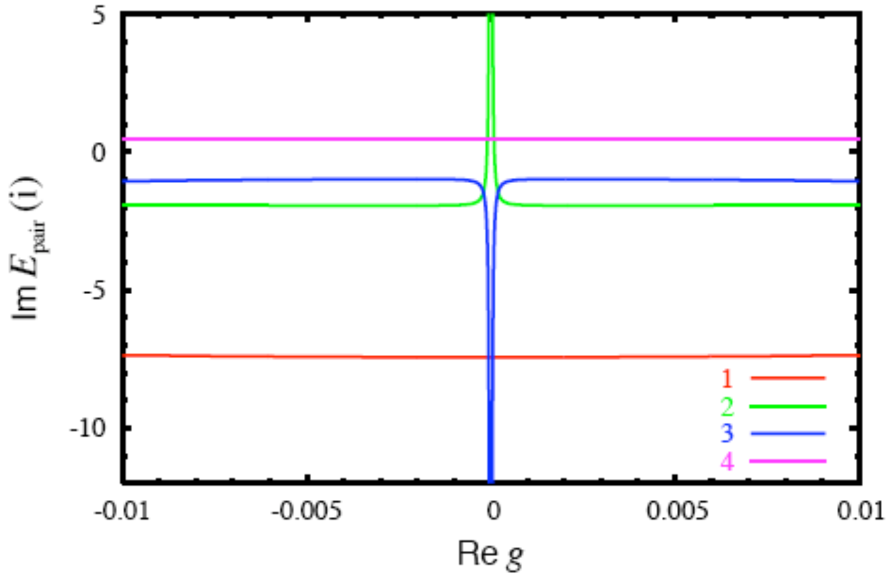
$\dim_H(g_i) = n - 1$

$\dim_H(g \neq g_i) = n$

Diabolic point arise from the 'fusion' of two exceptional points

$$E_{\text{pair}}(i) \equiv \langle i | g \sum_{kl} A_k^+ A_l | i \rangle$$





Singular behavior of $\langle i_1|\hat{O}|i_1\rangle, \langle i_2|\hat{O}|i_2\rangle$; Regular behavior of $\langle i_1|\hat{O}|i_1\rangle + \langle i_2|\hat{O}|i_2\rangle$

Conclusions

1. Degeneracies of levels in open quantum systems (OQS) appear even in the absence of symmetries
2. Different kinds of pole degeneracies in OQSs:
   - non-singular double-pole (integrable OQSs)
   - singular double-poles: exceptional points (integrable and non-integrable OQSs) and diabolic points (non-integrable OQSs)
3. Diabolic points correspond to local defects of the (Hilbert) space
4. Observable consequences of exceptional points - analogy with threshold effects.