Spectra and Symmetries of Nuclear Pairing

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Joint JUSTIPEN-LACM Meeting, March 2007

Quasi-Spin Algebra

$$\hat{S}_j^+ = \sum_{m>0} (-1)^{(j-m)} a^\dagger_{j\,m} a^\dagger_{j\,-m},$$
$$\hat{S}_j^- = \sum_{m>0} (-1)^{(j-m)} a_{j\,-m} a_{j\,m}$$

$$\hat{S}_j^0 = \frac{1}{2} \sum_{m>0} \left(a^\dagger_{j\,m} a_{j\,m} + a^\dagger_{j\,-m} a_{j\,-m} - 1, \right)$$

$$\hat{S}_j^0 = \hat{N}_j - \frac{1}{2}\Omega_j.$$

$\Omega_j = j + \frac{1}{2}$ = the maximum number of pairs that can occupy the level j

$$\hat{N}_j = \frac{1}{2} \sum_{m>0} \left(a^\dagger_{j\,m} a_{j\,m} + a^\dagger_{j\,-m} a_{j\,-m} \right).$$

$0 < \hat{N}_j < \Omega_j \longrightarrow \frac{1}{2}\Omega_j$ representation

- Nucleons interacting with a pairing force:

$$\hat{H} = \sum_{jm} \epsilon_j a^\dagger_{j\,m} a_{j\,m} - |G| \sum_{jj'} c_{jj'} \hat{S}^+_j \hat{S}^-_{j'} .$$

- When the pairing strength is separable ($c_{jj'} = c^*_j c_{j'}$):

$$\hat{H} = \sum_{jm} \epsilon_j a^\dagger_{j\,m} a_{j\,m} - |G| \sum_{jj'} c^*_j c_{j'} \hat{S}^+_j \hat{S}^-_{j'} .$$

- If we assume that the energy levels are degenerate the first term is a constant for a given number of pairs:

$$\hat{H} = -|G| \sum_{jj'} c^*_j c_{j'} \hat{S}^+_j \hat{S}^-_{j'} .$$

Other exactly solvable cases:

- Quasi-spin limit (all c_j's are the same):

$$\hat{H} = -|G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

- Richardson's solution:

$$\hat{H} = \sum_{jm} \epsilon_j a_{j\,m}^\dagger a_{j\,m} - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^- .$$

- Gaudin's model - somewhat different.

Define

$$\hat{S}^+(0) = \sum_j c_j^* \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(0) = \sum_j c_j \hat{S}_j^- ,$$

$$\hat{H} = -|G|\hat{S}^+(0)\hat{S}^-(0).$$

In the 1970's Talmi showed that under certain assumptions, a state of the form

$$\hat{S}^+(0)|0\rangle = \sum_j c_j^* \hat{S}_j^+ |0\rangle, \quad |0\rangle\text{: particle vacuum}$$

is an eigenstate of a class of Hamiltonians including the one above. Indeed

$$\hat{H}\hat{S}^+(0)|0\rangle = \left(-|G| \sum_j \Omega_j |c_j|^2 \right) \hat{S}^+(0)|0\rangle$$

What about other one-pair states?

For example for two levels j_1 and j_2, the orthogonal state

$$\left(\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^+_{j_1} - \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^+_{j_2} \right) |0\rangle,$$

is also an eigenstate with E=0.

Energy/$(-\lvert G\rvert)$	**State**
0	$\left(-\frac{c_{j_2}}{\Omega_{j_1}}\hat{S}^+_{j_1} + \frac{c_{j_1}}{\Omega_{j_2}}\hat{S}^+_{j_2}\right)\lvert 0\rangle$
$\Omega_{j_1}\lvert c_{j_1}\rvert^2 + \Omega_{j_1}\lvert c_{j_1}\rvert^2$	$\left(c^*_{j_1}\hat{S}^+_{j_1} + c^*_{j_2}\hat{S}^+_{j_2}\right)\lvert 0\rangle$

States with N=1 for two shells

What about other one-pair states?

For example for two levels j_1 and j_2, the orthogonal state

$$\left(\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^+_{j_1} - \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^+_{j_2} \right) |0\rangle,$$

is also an eigenstate with E=0.

Is there a systematic way to derive these states?

Yes, as showed by Pan, et al. for particle pair states.

Define

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}_j^- .$$

Then eigenstates are of the form

$$\hat{S}^+(x)\hat{S}^+(y)\cdots\hat{S}^+(z)|0\rangle$$

F. Pan, J.P. Draayer, W.E. Ormand, Phys. Lett. B **422**, 1 (1998)

Can we generalize this result to the cases where the shell is more than half full?

Yes!

A.B. Balantekin, J.H. de Jesus, and Y. Pehlivan, nucl-th/0702059

$$\hat{S}^+(x) = \sum_j \frac{c_j^*}{1 - |c_j|^2 x} \hat{S}_j^+ \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}_j^-$$

Introduce the operator

$$\hat{K}^0(x) = \sum_j \frac{1}{1/|c_j|^2 - x} \hat{S}_j^0$$

$$[\hat{S}^+(x), \hat{S}^-(0)] = [\hat{S}^+(0), \hat{S}^-(x)] = 2K^0(x)$$

$$[\hat{K}^0(x), \hat{S}^\pm(y)] = \pm \frac{\hat{S}^\pm(x) - \hat{S}^\pm(y)}{x - y}$$

This is very similar to Gaudin algebra!

$$\hat{S}^+(0)\hat{S}^+(z_1^{(N)})\ldots\hat{S}^+(z_{N-1}^{(N)})|0\rangle$$

is an eigenstate if the following Bethe ansatz equations are satisfied:

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}} \qquad m = 1, 2, \ldots N-1.$$

$$E_N = -|G| \left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}} \right)$$

Pan et al did not note but this is an eigenstate if the shell is at most half full.

Similarly

$$\hat{S}^+(x_1^{(N)})\hat{S}^+(x_2^{(N)})\ldots\hat{S}^+(x_N^{(N)})|0\rangle$$

is an eigenstate with zero energy if the following Bethe ansatz equations are satisfied:

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - x_m^{(N)}} = \sum_{k=1(k\neq m)}^{N} \frac{1}{x_m^{(N)} - x_k^{(N)}} \quad \text{for every} \quad m = 1, 2, \ldots, N$$

Again this is a state if the shell is at most half full.

What if the available states are more than half full? There are degeneracies:

No. of Pairs	**Energy/$(-\|G\|)$**	**State**
1	$\sum_j \Omega_j \|c_j\|^2$	$\hat{S}^+(0)\|0\rangle$
N_{max}	$\sum_j \Omega_j \|c_j\|^2$	$\|\bar{0}\rangle$

$|0\rangle$: particle vacuum

$|\bar{0}\rangle$: state where all levels are completely filled

If the shells are more than half full then the state

$$\hat{S}^-(z_1^{(N)})\hat{S}^-(z_2^{(N)})\dots\hat{S}^-(z_{N-1}^{(N)})|\bar{0}\rangle$$

is an eigenstate with energy

$$E = -G\left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}}\right)$$

if the following Bethe ansatz equations are satisfied

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}}$$

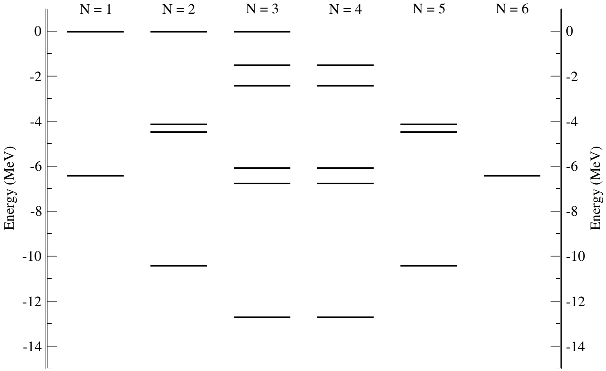
Here $N_{max} + 1 - N =$ number of particle pairs

Particle-hole degeneracy:

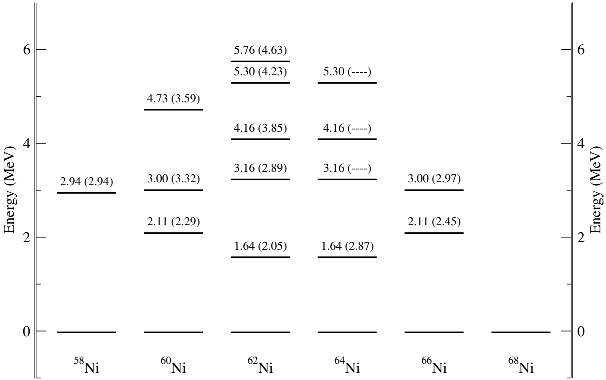
No. of Pairs	**State**
N	$\hat{S}^+(0)\hat{S}^+(z_1^{(N)})\ldots\hat{S}^+(z_{N-1}^{(N)})\|0\rangle$
$N_{max}+1-N$	$\hat{S}^-(z_1^{(N)})\hat{S}^-(z_2^{(N)})\ldots\hat{S}^-(z_{N-1}^{(N)})\|\bar{0}\rangle$

$$E = -G\left(\sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}}\right)$$

$$\sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_m^{(N)} - z_k^{(N)}}$$



Results for the sd shell with $0d_{5/2}$, $0d_{3/2}$, and $1s_{1/2}$



theory (experiment)

Solutions of Bethe Ansatz equations

$$x_i^{(N)} = \frac{1}{|c_{j_2}|^2} + \eta_i^{(N)} \left(\frac{1}{|c_{j_1}|^2} - \frac{1}{|c_{j_2}|^2} \right)$$

$$\sum_{k=1(k\neq i)}^{N} \frac{1}{\eta_i^{(N)} - \eta_k^{(N)}} - \frac{\Omega_{j_2}/2}{\eta_i^{(N)}} + \frac{\Omega_{j_1}/2}{1-\eta_i^{(N)}} = 0$$

In 1914 Stieltjes showed that the polynomial

$$p_N(z) = \prod_{i=1}^{N} (z - \eta_i^{(N)})$$

satisfies the hypergeometric equation

$$z(1-z)p_N'' + \left[-\Omega_{j_2} + \left(\Omega_{j_1}\Omega_{j_2}\right) z\right] p_N' + N\left(N - \Omega_{j_1} - \Omega_{j_2} - 1\right) p_N = 0$$

Conclusions

- We showed that the Bethe ansatz technique can be applied to the nuclear pairing Hamiltonian with separable pairing strengths and degenerate energy levels in a purely algebraic fashion.
- The eigenstates with N pairs of nucleons where $N \leq N_{max}/2$ and the eigenstates with $N_{max} + 1 - N$ pairs of nucleons have the same energy and these states can be found by solving the same equations of Bethe ansatz (except for the zero energy states).
- It may be possible to incorporate non-degenerate energy levels perturbatively.